Test 1 Numerical Mathematrics 2
November, 2018

Duration: 1 hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test.

1. Consider the quadratic equation $x^2 - 2x + d = 0$, where d is the given data.

   (a) [1.5] Compute a bound for the absolute and relative condition number of this problem for the case $|d| < 1/2$ (also any perturbed d satisfies this condition). (The relation $\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$ might be useful.)

   > **Solution:** Using the abc-formula one readily finds that $x(d) = 1 \pm \sqrt{1-d}$. Now (note $\pm$ is strict below)
   >
   > $$\begin{aligned} \delta x = x(d+\delta d) - x(d) &= 1 \pm \sqrt{1-(d+\delta d)} - (1 \pm \sqrt{1-d}) \\ &= \pm(\sqrt{1-(d+\delta d)} - \sqrt{1-d}) \\ &= \pm \frac{\delta d}{\sqrt{1-(d+\delta d)} + \sqrt{1-d}} \end{aligned}$$
   >
   > The minimum value of the denominator is for the range of d considered $2\sqrt{1/2} = \sqrt{2}$. So $K_{abs}(d) = \max_{\{\delta d \ | \ \ |d+\delta d|, |d|<1/2\}} |\delta x|/|\delta d| < 1/\sqrt{2}$.
   > The relative condition number is just the absolute condition number times $|d|/|x|$: $K_{rel} < |d|/[\sqrt{2}(1 \pm \sqrt{1-d})]$.

   (b) [1.5] Compute the absolute condition number at $d = 1$ for small perturbations of d.

   > **Solution:** We can just differentiate x wrt d and evaluate the result at $d = 1$ to get the result. The derivative is
   >
   > $$dx/dd = - \pm \frac{1}{2\sqrt{1-d}}$$
   >
   > We see that K_{abs} goes to infinity for $d \to 1$.
   > We could also use the previous part which immediately leads to $\delta x = \sqrt{\delta d}$ which still says that δx goes to zero with δd. However, the rate $\delta x/\delta d$ goes to infinity if δd goes to zero, which can be seen by dividing on both sides by δd.

2. Consider an algorithm which computes x from d, i.e. $x = G(d)$. However, due to round-off errors we are computing $\hat{x} = \hat{G}(d, u)$ where u is the round-off unit and $G(d) = \hat{G}(d, 0)$.

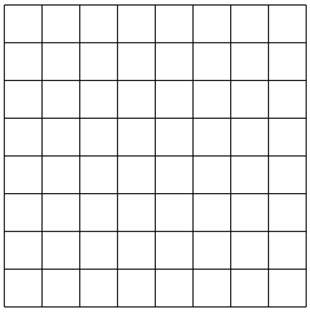
   (a) [0.5] How is the stable relative backward error defined here?

   > **Solution:** $\hat{x}$ is the solution of $\hat{x} = G(d + \delta d)$ where $||\delta d||/||d|| < Cu$.

   (b) [0.5] How can one find the relative forward error from the relative backward error?

Solution: From $x = G(d)$ one hopes to find a finite K_{rel} s.t. $||\delta x||/||x|| < K_{rel}||\delta d||/||d||$. Using the backward error we now know that the forward error is $||\hat{x} - x||/||x|| < K_{rel}Cu$.

3. Consider the graph



(a) [1.5] Copy this graph to the paper and use that to explain the nested dissection ordering on this graph. You may limit the nesting to two steps. Also make a sketch of the associated vector of unknowns and the associated matrix structure such that it is clear where the unknowns and associated equations go.

Solution: Sketch attached.

(b) [0.5] Explain the relevance of reordering the matrix A, corresponding to the graph, for solving $Ax = b$

Exam questions continue on other side

4. Consider the problem

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

(a) [1.5] Show that the pseudo-inverse of the matrix is given by

$$\begin{bmatrix} 1/14 & 3/10 \\ 1/7 & 0 \\ 3/14 & -1/10 \end{bmatrix}$$

Solution: We just have to compute the SVD of A, i.e. $A = U\Sigma V^T$. From this follows that $A^T A = V\Sigma^T \Sigma V$ and $AA^T = U\Sigma\Sigma^T U$.

$$AA^T = \begin{bmatrix} 14 & 0 \\ 0 & 10 \end{bmatrix} \equiv D^2$$

So the singular values are $\sqrt{14}$ and $\sqrt{10}$ and $U = I$ From $A = U\Sigma V^T$ it follows that $A = \Sigma V^T$. So the first two columns of V are just the normalized rows of A. Now

$$A^\dagger = V\Sigma^\dagger = (D^{-1}A)^T D^{-1} = A^T D^{-2}$$

(b) [1] Solve the problem using the pseudo-inverse.

Solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/14 & 3/10 \\ 1/7 & 0 \\ 3/14 & -1/10 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/7 + 3/5 \\ 6/7 \\ 9/7 - 1/5 \end{bmatrix} = \begin{bmatrix} 36/35 \\ 6/7 \\ 38/35 \end{bmatrix}$$

(c) [0.5] What is the central idea of using the pseudo-inverse?

Solution: The equation above admits a whole class of solutions. From that it selects that solution that is shortest in 2-norm.